

1. Two point charges are fixed on the xy-plane. The charge $+Q$ is placed at the point $(-d, 0)$, and the charge $+4Q$ is placed at the point $(d, 0)$. Express your results using unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

[7 Pts.]

- (a) Calculate the forces $\mathbf{F}_Q$ and $\mathbf{F}_{4Q}$ acting on charges.

[8 Pts.]

- (b) Find the expression for the electric field $\mathbf{E}$ along the y-axis ($x = 0$).

[7 Pts.]

- (c) Determine the potential energy of the entire configuration.

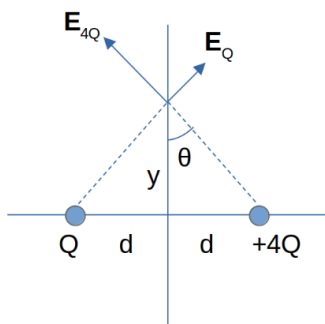
[8 Pts.]

- (d) We now want to place a third charge $-Q$ between the two charges on the x-axis (for $-d < x < d$). Where should it be placed so that the net force on $-Q$ is zero?

a)

$$\mathbf{F}_{4Q} = \frac{1}{4\pi\epsilon_0} \frac{4Q^2}{4d^2} \hat{\mathbf{i}} = \frac{Q^2}{4\pi\epsilon_0 d^2} \hat{\mathbf{i}}$$

$$\mathbf{F}_Q = -\frac{Q^2}{4\pi\epsilon_0 d^2} \hat{\mathbf{i}}$$



b)

$$|\mathbf{E}_{4Q}| = \frac{1}{4\pi\epsilon_0} \frac{4Q}{d^2 + y^2} \quad \& \quad |\mathbf{E}_Q| = \frac{1}{4\pi\epsilon_0} \frac{Q}{d^2 + y^2}$$

$$\sin \theta = \frac{d}{(d^2 + y^2)^{1/2}}, \quad \cos \theta = \frac{y}{(d^2 + y^2)^{1/2}}$$

$$\Rightarrow \mathbf{E}_{4Q} = \frac{Q}{4\pi\epsilon_0} \left(\frac{-4d}{(d^2 + y^2)^{3/2}} \hat{\mathbf{i}} + \frac{4y}{(d^2 + y^2)^{3/2}} \hat{\mathbf{j}} \right)$$

$$\mathbf{E}_Q = \frac{Q}{4\pi\epsilon_0} \left(\frac{d}{(d^2 + y^2)^{3/2}} \hat{\mathbf{i}} + \frac{y}{(d^2 + y^2)^{3/2}} \hat{\mathbf{j}} \right)$$

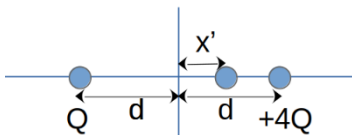
Thus, the electric field along the y-axis is

$$\mathbf{E} = \mathbf{E}_{4Q} + \mathbf{E}_Q = \frac{Q}{4\pi\epsilon_0 (d^2 + y^2)^{3/2}} (-3d\hat{\mathbf{i}} + 5y\hat{\mathbf{j}})$$

c)

$$U = \frac{1}{4\pi\epsilon_0} \frac{4Q^2}{2d} = \frac{1}{2\pi\epsilon_0} \frac{Q^2}{d}$$

d)



$$\mathbf{F} = \mathbf{F}'$$

$$\Rightarrow \frac{-4Q^2}{4\pi\epsilon_0 (d-x)^2} = \frac{-Q^2}{4\pi\epsilon_0 (d+x)^2}$$

$$4(d-x)^2 = (d+x)^2$$

$$x = \frac{-d}{3}$$

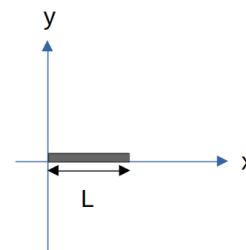
Position of $-Q$ charge is $(\frac{-d}{3}, 0)$

2. A thin rod of length L is placed along the x-axis as shown in the figure. The rod carries a uniformly distributed charge Q .

[10 Pts.] (a) Determine the electric potential $V(x)$ for the points $x > L$ along the x-axis.

[10 Pts.] (b) Determine the electric potential $V(y)$ for the points $y > 0$ along the y-axis.

[15 Pts.] (c) Find the expression for the y-component of the electric field along the y-axis.

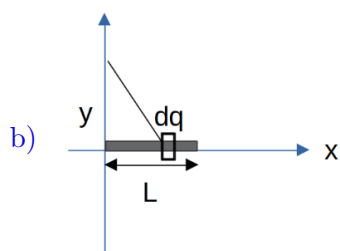


Hint: $\int \frac{du}{u^2+a^2} = \frac{1}{a} \tan^{-1}(\frac{u}{a}) + C$, $\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u + \sqrt{u^2+a^2}) + C$, $\int \frac{du}{(u^2 \pm a^2)^{3/2}} = \frac{\pm u}{a^2 \sqrt{u^2 \pm a^2}} + C$

a) Charge density is given as

$$dq = \frac{Q dx'}{L}$$

$$\begin{aligned} \text{Then, } V(x) &= \frac{1}{4\pi\epsilon_0} \int \frac{dq}{x - x'} = \frac{Q}{4\pi\epsilon_0 L} \int_0^L \frac{dx'}{x - x'} \\ \Rightarrow V(x) &= \frac{Q}{4\pi\epsilon_0 L} \ln\left(\frac{x}{x-L}\right) \end{aligned}$$



$$V(y) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{\sqrt{x'^2 + y^2}} = \frac{Q}{4\pi\epsilon_0 L} \int_0^L \frac{dx'}{\sqrt{x'^2 + y^2}}$$

by using the proper integral from the hint

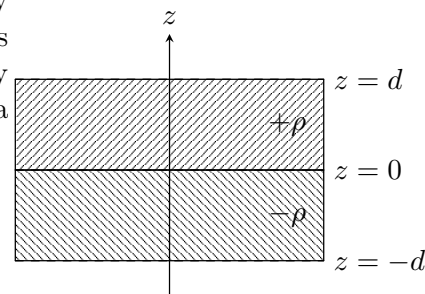
$$\begin{aligned} V(y) &= \frac{Q}{4\pi\epsilon_0 L} \ln(x' + \sqrt{x'^2 + y^2}) \Big|_0^L \\ \Rightarrow V(y) &= \frac{Q}{4\pi\epsilon_0 L} \ln\left(\frac{L + \sqrt{L^2 + y^2}}{y}\right) \end{aligned}$$

c)

$$\begin{aligned} E_y &= -\frac{\partial V}{\partial y} = -\frac{Q}{4\pi\epsilon_0 L} \left(\frac{y}{\sqrt{L^2 + y^2}} \frac{1}{L + \sqrt{L^2 + y^2}} - \frac{1}{y} \right) \\ &= \frac{Q}{4\pi\epsilon_0 L y} \frac{L^2 + L\sqrt{L^2 + y^2}}{L^2 + y^2 + L\sqrt{L^2 + y^2}} \\ \Rightarrow E_y &= \frac{Q}{4\pi\epsilon_0 y} \frac{L + \sqrt{L^2 + y^2}}{L^2 + y^2 + L\sqrt{L^2 + y^2}} = \frac{Q}{4\pi\epsilon_0 y} \frac{1}{\sqrt{L^2 + y^2}} \end{aligned}$$

Checking: $\lim_{\frac{L}{y} \rightarrow 0} E_y = \frac{Q}{4\pi\epsilon_0 L y^2}$ gives point particle electric field as expected

3. Consider two infinite nonconducting slabs of uniform charge density placed symmetrically around the $z = 0$ plane: The first slab occupies the region $0 < z < d$ and carries a uniform positive charge density ρ . The second slab occupies the region $-d < z < 0$ and carries a uniform negative charge density $-\rho$.



(a) Find the electric field $\mathbf{E}(z)$ in all three regions:

- [5 Pts.] i. Above the slabs: $z > d$
 [5 Pts.] ii. Inside slabs: $-d < z < d$
 [5 Pts.] iii. Below the slabs: $z < -d$

[5 Pts.] (b) Sketch (plot) how the electric field varies as a function of z .

[5 Pts.] (c) Determine the location(s) where the magnitude of the electric field is maximum.

[10 Pts.] (d) Compute the potential difference between a point above the upper surface ($z = d$) and a point below the lower surface ($z = -d$).

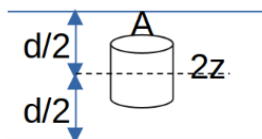
a)

For a single slab with center at $z = 0$

$$\oint \mathbf{E} \cdot d\mathbf{A} = 2A|\mathbf{E}| = \frac{2zA\rho}{\epsilon_0} \quad \text{inside}$$

$$\mathbf{E} = \frac{\rho}{\epsilon_0} z \hat{\mathbf{k}} \quad \text{inside} \quad |z| < \frac{d}{2}$$

$$\mathbf{E} = \frac{\rho}{\epsilon_0} \frac{d}{2} \text{sign}(z) \hat{\mathbf{k}} \quad \text{outside} \quad |z| > \frac{d}{2}$$



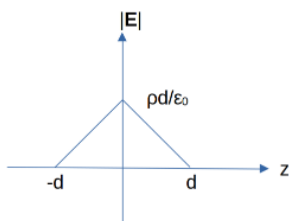
two of these result in

i. $z > d$ Fields cancel each other $\mathbf{E} = 0$

ii. $\mathbf{E} = \frac{\rho}{\epsilon_0} (d - |z|) (-\hat{\mathbf{k}})$

iii. $z < -d$ Fields cancel each other $\mathbf{E} = 0$

b) $\mathbf{E} = |\mathbf{E}|(-\hat{\mathbf{k}})$



c) maximum at $z = 0$, $|\mathbf{E}| = \frac{\rho d}{\epsilon_0}$

d) Once $|\Delta V| = -\int \mathbf{E} \cdot d\mathbf{l}$,

$$|\Delta V| = \int_{-d}^d |\mathbf{E}(z)| dz = \frac{\rho d}{\epsilon_0} d = \frac{\rho d^2}{\epsilon_0} \quad \text{upper region at a higher potential}$$